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ACCUARCY OF INTERPOLATED AND DIFFERENTIATED SATELLITE POSITIONS

Ted Sims

Naval Surface Weapons Center
Dahlgren, Virginia

October 1974

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October 1974

ACCURACY OF INTERPOLATED AND DIFFERENTIATED SATELLITE POSITIONS

Ted Sims
Warfare Analysis Department

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FOREWORD

This report presents methods of reducing the amount of computer storage required to perform the computation of satellite orbits. A factor of six reduction will be realized by changing current practice at the Naval Surface Weapons Center/Dahlgren Laboratory to that suggested here.

The study was performed in support of DoD activities utilizing near earth satellites.

The report was reviewed by R. J. Anderle, Head, Astronautics and Geodesy Division, Naval Surface Weapons Center/Dahlgren Laboratory.

RELEASED BY:

Ralph A. Niemann

R. A. NIEMANN, Head
Warfare Analysis Department

ABSTRACT

This study was undertaken to demonstrate the feasibility of storing just the position segment of the satellite ephemeris at maximum intervals of time. Position of the satellite at times not on the ephemeris would be found by interpolation. Satellite velocity and acceleration would be calculated via numerical differentiation of the positions. Partial derivatives of satellite velocity with respect to orbit parameters would be obtained from the numerical time derivatives of the position partials with respect to orbit parameters. Satellites at three specific altitudes were considered: (1) a low altitude satellite (160 km), (2) a medium altitude (960 km), (3) a high altitude satellite (12000 km). To achieve acceptable accuracy standards, the optimum interpolation order and storage interval as a function of satellite altitude are as follows:

Satellite Altitude	Interpolation Interval	Storage Interval
low	08	90 sec
med	08	180 sec
high	08	900 sec

TABLE OF CONTENTS

INTRODUCTION

Given the initial position and velocity vector of a satellite at some time T, a Cowell integration routine is used to generate position and velocity components at some time later than the initial time. The integration interval used is a function of the satellite altitude, with low altitude satellites being integrated at smaller intervals because the lower satellites are more sensitive to irregularities in the earth's gravitational field. See Reference 1.

Table 1 shows how the Naval Surface Weapons Center/Dahlgren Laboratory stores the satellite epnemeris. Position, velocity, acceleration, and partial derivatives of position and velocity with respect to oribt parameters are given at each integration time step. The satellite velocity and acceleration are time derivatives of the satellite position. The partial derivatives of velocity with respect to orbit parameters are time derivatives of the respective position partials with respect to orbit parameters, i.e.

$$\frac{\partial \dot{x}}{\partial \rho} = \frac{d}{dt} \frac{\partial x}{\partial \rho}$$

Thus, if the position ephemeris is defined to be that part of the complete ephemeris containing satellite positions and partial derivatives of position with respect to orbit parameters, then in theory it is possible to reconstruct the complete ephemeris from the position ephemeris. It is the purpose of this report to determine exactly this. Can the position ephemeris, stored at an interval of time KT, where T is the integration interval and K is the positive integer, be used to reconstruct the complete ephemeris within acceptable (accuracy) tolerances?

T A B L E 1

Astro Satellite Trajectory Tape (B1 Tape)

b. Block #2, 3, 4 etc.

Input: Binary Word #	Input: Style	Data Type	Output: Line #	Output: Word #	Output: Item
		Tape number	1	1	9
1	I	Number of words		2	J
2	I	Consecutive block # (starting with 2)		3	K
3	F	$t - t_o$ (in seconds)	2	1	TRA(1)
4	F	x		2	TRA(2)
5	F	y		3	TRA(3)
6	F	z		4	TRA(4)
7	F	$\dot{x}$		5	TRA(5)
8	F	$\dot{y}$		6	TRA(6)
9	F	$\dot{z}$	3	1	TRA(7)
10	F	$\ddot{x}$		2	TRA(8)
11	F	$\ddot{y}$		3	TRA(9)
12	F	$\ddot{z}$		4	TRA(10)
13	F	$\frac{\partial x(t)}{\partial P_1(t_o)}$		5	TRA(11)
14	F	$\frac{\partial y(t)}{\partial P_1(t_o)}$		6	TRA(12)
15	F	$\frac{\partial z(t)}{\partial P_1(t_o)}$	4	1	TRA(13)
16	F	$\frac{\partial \dot{x}(t)}{\partial P_1(t_o)}$		2	TRA(14)
17	F	$\frac{\partial \dot{y}(t)}{\partial P_1(t_o)}$		3	TRA(15)
18	F	$\frac{\partial \dot{z}(t)}{\partial P_1(t_o)}$		4	TRA(16)
. . . .	. . . .	The partial derivatives indicated by line 3 (words 5 and 6) and line 4 (words 1,2,3 and 4) are repeated for each parameter to be altered (maximum of 22 parameters). The number of parameters, ITR, appears in block #1.		. . . . 6 words per line	. . . . TRA(J)

PROCEDURE

Initial conditions for three representative satellites of differing altitudes were integrated forward and a complete ephemeris was generated for each. Two trajectories were integrated for the low and medium altitude satellites. The low altitude satellite at 15 and 30 seconds. The medium altitude satellite at 30 and 60 seconds. The integration interval for the high altitude satellite was 60 seconds. Table 2 shows the orbit parameters for each satellite

The extended storage interval KT, is obtained by sampling every K^{th} integration time line from the nominal ephemeris. In order to reconstruct the ephemeris to its original denseness, i.e. a write interval equal to the integration interval, a standard centered Lagrange interpolation formula of the form

$$f(t) = \sum_{K=-m}^{m+1} A_K(s) f_K \qquad (1)$$

is used. The order, M, of the interpolation routine is equal to 2m + 2. Satellite velocity and acceleration are obtained from the position segment of the ephemeris by differentiating Equation (1) with respect to t once and twice, respectively. See Appendix A for the mathematical formulation. In similar fashion the position partials are reconstructed to their original denseness using Equation (1), and the velocity partials are reconstructed by using the position partials and the time derivative of Equation (1). The storage interval KT and the interpolation order M were varied for each satellite until a K and M were found such that the complete ephemeris could be reproduced from the position segment to within the following tolerances (in any one component): position; 1.0×10^{-4} km. Velocity; 1.5×10^{-6} km/sec. Acceleration; 1.5×10^{-8} km/sec^2.

As mentioned above the initial conditions of the low and medium altitude satellites were integrated at two different time intervals. For example, consider the low altitude satellite and an extended storage interval of 90 seconds. If the trajectory is integrated at a 30 second time step, no comparison can be made between reconstructed ephemeris values and the actual integrated values at the half interval value of 45 seconds. Thus the necessity of having a 15 second integration interval. On the other hand with a storage interval of 120 seconds, a 30 second integration interval will allow comparison at half step storage intervals.

Tables 3, 4, 5, show the results of position, velocity, and acceleration recovery as a function of storage interval and interpolation order. The R_I column shows the maximum deviation that occurred between

TABLE 2

SATELLITE	PERIOD (min)	INCLINATION (°)	APOGEE HEIGHT (nm)	PERIGEE HEIGHT (nm)
Low	88.66	95.20	147.02	74.31
Medium	106.93	89.25	603.98	563.08
High	469.14	125.00	7350.26	7350.26

T A B L E 3

LOW ALTITUDE SATELLITE

INTERPOLATION ORDER	STORAGE INTERVAL	MAXIMUM DIFFERENCE IN ANY ONE COORDINATE				
[redacted]	[redacted]	R_I	$\dot{R}_D$	$\ddot{R}_D$	$\dot{R}_I$	$\ddot{R}_I$
08	60	$1.2\text{x}10^{-6}$	$7.\text{x}10^{-8}$	$2.9\text{x}10^{-9}$	$3.0\text{x}10^{-8}$	$8.\text{x}10^{-10}$
06	90	$7.\ \text{x}10^{-5}$	$3.\text{x}10^{-6}$	$8.\ \text{x}10^{-8}$	$3.0\text{x}10^{-7}$	$8.\text{x}10^{-9}$
08	90	$1.0\text{x}10^{-5}$	$4.\text{x}10^{-7}$	$1.2\text{x}10^{-8}$	$2.4\text{x}10^{-7}$	$6.\text{x}10^{-9}$
10	90	$7.\ \text{x}10^{-6}$	$2.5\text{x}10^{-7}$	$8.0\text{x}10^{-9}$	$1.7\text{x}10^{-7}$	$4.\text{x}10^{-9}$
08	120	$6.2\text{x}10^{-5}$	$1.53\text{x}10^{-6}$	$4.2\text{x}10^{-8}$	$1.1\text{x}10^{-6}$	$2.8\text{x}10^{-8}$
10	120	$3.\ \text{x}10^{-5}$	$1.\ \text{x}10^{-6}$	$2.\ \text{x}10^{-8}$	$7.\ \text{x}10^{-7}$	$1.7\text{x}10^{-8}$
12	120	$2.9\text{x}10^{-5}$	$8.\ \text{x}10^{-7}$	$1.7\text{x}10^{-8}$	$7.\ \text{x}10^{-7}$	$1.9\text{x}10^{-8}$
18	180	$2.\ \text{x}10^{-4}$	$4.\ \text{x}10^{-6}$	$6.\ \text{x}10^{-8}$	$3.7\text{x}10^{-6}$	$6.\ \text{x}10^{-8}$

TOLERANCE

R : $1.\ \text{x}\ 10^{-4}$ km

$\dot{R}$: $1.5\ \text{x}\ 10^{-6}$ km/sec

$\ddot{R}$: $1.5\ \text{x}\ 10^{-8}$ km/sec^2

T A B L E 4

MEDIUM ALTITUDE SATELLITE

INTERPOLATION ORDER	STORAGE INTERVAL	MAXIMUM DIFFERENCE IN ANY ONE COORDINATE				
[redacted]		R_I	$\dot{R}_D$	$\ddot{R}_D$	$\dot{R}_I$	$\ddot{R}_I$
08	120	$3.\times10^{-6}$	2.2×10^{-7}	1.5×10^{-9}	1.2×10^{-7}	$8.\times10^{-10}$
08	180	3.2×10^{-5}	$7.\times10^{-7}$	$1.\times10^{-8}$	$4.\times10^{-7}$	3.7×10^{-9}
08	240	2.4×10^{-4}	3.7×10^{-6}	$4.\times10^{-8}$	1.2×10^{-6}	1.5×10^{-8}
10	240	$1.\times10^{-4}$	1.2×10^{-6}	1.6×10^{-8}	$1.\times10^{-6}$	1.2×10^{-8}
12	240	$8.\times10^{-5}$	$8.\times10^{-7}$	1.2×10^{-8}	$8.\times10^{-7}$	1.2×10^{-8}

TOLERANCE

R : 1.0×10^{-4} km

$\dot{R}$: 1.5×10^{-6} km/sec

$\ddot{R}$: 1.5×10^{-8} km/sec^2

TABLE 5

HIGH ALTITUDE SATELLITE

INTERPOLATION ORDER	STORAGE INTERVAL	MAXIMUM DIFFERENCE IN ANY ONE COORDINATE				
[redacted]		R_I	$\dot{R}_D$	$\ddot{R}_D$	$\dot{R}_I$	$\ddot{R}_I$
08	840	$4.\times10^{-5}$	1.5×10^{-7}	$5.\times10^{-10}$	$8.\times10^{-9}$	$3.\times10^{-12}$
08	900	$7.\times10^{-5}$	2.7×10^{-7}	$8.\times10^{-10}$	1.3×10^{-8}	$6.\times10^{-12}$
08	960	1.1×10^{-4}	$3.\ \times10^{-7}$	$1.\times10^{-9}$	2.2×10^{-8}	$1.\times10^{-11}$
08	1080	2.6×10^{-4}	$8.\times10^{-7}$	2.1×10^{-9}	$6.\ \times10^{-8}$	2.2×10^{-11}
08	1200	$6.\ \times10^{-4}$	1.8×10^{-6}	$4.\ \times10^{-9}$	1.3×10^{-7}	$5.\ \times10^{-11}$
08	1420	2.5×10^{-3}	$6.\ \times10^{-6}$	1.2×10^{-8}	$6.\ \times10^{-7}$	2.1×10^{-10}

TOLERANCE

R : $1.\ \times 10^{-4}$ km

$\dot{R}$: 1.5×10^{-6} km/sec

$\ddot{R}$: 1.5×10^{-8} km/sec^2

any position component when the interpolated position was differenced with the integrated position. Likewise $\dot{R}_D$ and $\ddot{R}_D$ show maximum deviation for velocity and acceleration components obtained from the position components. Lastly $\dot{R}_I$ and $\ddot{R}_I$ show the maximum deviation in any one component between interpolated velocity and acceleration obtained using the designated storage interval and the value of velocity and acceleration from the nominal complete ephemeris. All maximum deviations are for a period of time covering more than one revolution of the satellite.

Low Altitude Satellite

The optimum interpolation order and storage interval for the low altitude satellite is (from Table 3) 8^{th} order and 90 seconds. Accurate retrieval cannot be made using a 120 second storage interval even with a 12^{th} order routine. Notice also that in almost all cases the interpolated velocities and accelerations are more accurate than those obtained by numerical differentiation of the position components.

Medium Altitude Satellite

The optimum case for the medium altitude satellite is the 8^{th} order routine with a 180 second storage interval. Here a 12^{th} order routine and a 240 second interval passes the required tolerances. However it was felt that the computer time necessary for retrieval using this higher order outweighs the storage advantage of the longer interval.

High Altitude Satellite

The optimum interpolation order and storage interval for the high altitude satellite is 8^{th} order and 900 seconds. Unlike the case of the low altitude satellite, the high altitude satellite shows considerably more accuracy in interpolated velocity and acceleration components than their corresponding differentiated counterparts.

Appendix B contains plots of the resulting differences using various storage intervals and interpolated orders for each satellite. For each interpolation order and storage interval there is a set of five plots. Interpolated position versus nominal position; differentiated and twice differentiated position differences giving velocity and acceleration differences; and interpolated velocity and acceleration versus nominal values. Under each title is the following key (XX ORD YYY SEC ZZH). XX gives the order of the interpolation; YYY the storage interval and ZZH contains the integration step of the nominal trajectory in seconds plus the altitude of the satellite. Thus 10 ORD 180 SEC 30 M would identify the medium altitude satellite with the nominal trajectory integrated at 30 seconds, a 10^{th} order routine and a storage interval of 180 seconds.

The next area of consideration involves the partial derivatives. We attempt to determine to what degree of accuracy the partial derivatives can be reconstructed from the position ephemeris using the optimum interpolation order and storage interval determined above for each satellite. Again the position partials are reconstructed at the nominal write interval using Equation (1), and the velocity partials are reconstructed using the position partials and the time derivative of Equation (1).

The resultant reconstructed position partial derivatives were normalized and compared to nominal values by computing the following difference,

$$\left(\frac{\partial X^2}{\partial P_K} + \frac{\partial Y^2}{\partial P_K} + \frac{\partial Z^2}{\partial P_K}\right)^{\frac{1}{2}}_{\text{INTER}} - \left(\frac{\partial X^2}{\partial P_K} + \frac{\partial Y^2}{\partial P_K} + \frac{\partial Z^2}{\partial P_K}\right)^{\frac{1}{2}}$$

where the radical subscripted by INTER represents interpolated values and the entries within the second radical, nominal values. The velocity partials were compared in like manner

$$\left(\frac{\partial \dot{X}^2}{\partial P_K} + \frac{\partial \dot{Y}^2}{\partial P_K} + \frac{\partial \dot{Z}^2}{\partial P_K}\right)^{\frac{1}{2}}_{\text{DIFF}} - \left(\frac{\partial \dot{X}^2}{\partial P_K} + \frac{\partial \dot{Y}^2}{\partial P_K} + \frac{\partial \dot{Z}^2}{\partial P_K}\right)^{\frac{1}{2}} .$$

Table 6 lists the maximum differences between reconstructed normalized partials and their normalized nominal counterparts. 3 UNITS_8 indicates a maximum deviation of 3 units in the eight significant place. Maximum deviations for satellite positions other than near polar or equatorial regions were found to be much less than the values given in Table 6.

In an attempt to give physical significance to the entries of Table 6, the following test was conducted. In the case of the low altitude satellite two propogated trajectories were constructed. (A propogated trajectory is constructed using the partial derivatives of position and velocity with respect to orbit parameters and a set of improvements to the orbital elements to make corrections to the original trajectory.) The first propogated trajectory was formed using the nominal position ephemeris written at a 15 second interval and a representative set of changes in orbital parameters. The second propogated trajectory for the low altitude satellite was formed using the nominal low altitude ephemeris written at 90 seconds plus the same set of representative changes to orbital parameters. An 8^{th} order Lagrangian interpolation routine was used on this trajectory to write position partials every 15 seconds. Figure 1 shows the total position deviation between these two propogated trajectories. The maximum difference is 4 cm.

TABLE 6

SATELLITE	MAXIMUM DEVIATION OF NORMALIZED PARTIAL DERIVATIVES			
[redacted]	Position Partials		Velocity Partials	
	Equator	Pole	Equator	Pole
Low	3 Units$_8$	1.5 Units$_8$	25 Units$_7$	8 Units$_8$
Medium	7 Units$_8$	18 Units$_8$	13 Units$_7$	21 Units$_7$
High	7 Units$_8$	6 Units$_8$	5 Units$_7$	19 Units$_7$

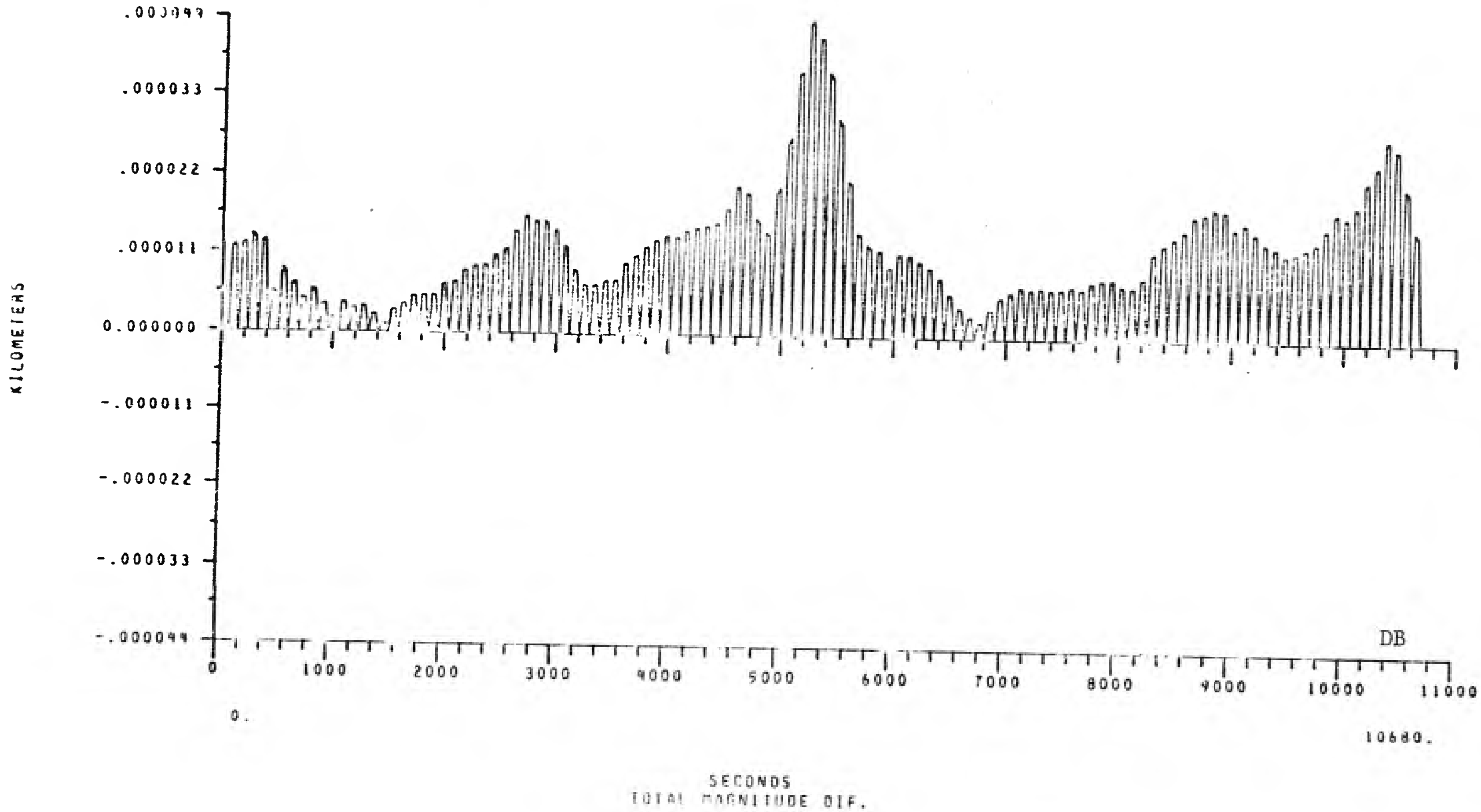
FIGURE 1

TRAJECTORY COMPARISON PLOT

REV NO. 0 IS COMPARED TO REV NO. 0

EPOCH (YR,DAY,SEC)= 73 316 81720

SATELLITE= 78



In like manner two propogated trajectories were created for the medium altitude satellite. The first was computed using the nominal medium altitude trajectory written at 60 seconds and a representative set of changes to orbital parameters. The second propogated trajectory used the nominal medium altitude trajectory written at 180 seconds. Figure 2 shows the total difference between these two trajectories. Maximum deviation is below the 1 cm level.

In each case the difference between the two trajectories is small adding credence to the fact that the interpolation orders and storage intervals listed above are indeed optimum not only for position, velocity and acceleration but also for partial derivatives.

As a result of the above findings it is now possible to store just the position segment of the ephemeris (i.e. satellite position and partial derivatives of position with respect to dynamic parameters at a storage interval greater than the integration interval. This results in a tremendous savings in storage requirements. Omitting velocity components and partial derivatives of velocity with respect to dynamic parameters results in storage requirements being reduced by a factor of 2. In the case of the low and medium altitude satellites' ephemeris, presently stored at 30 seconds and 60 seconds, respectively, storing the position ephemeris at the optimum storage intervals results in total storage requirements being reduced by a factor of 6. Even greater savings result in the case of the high altitude satellite.

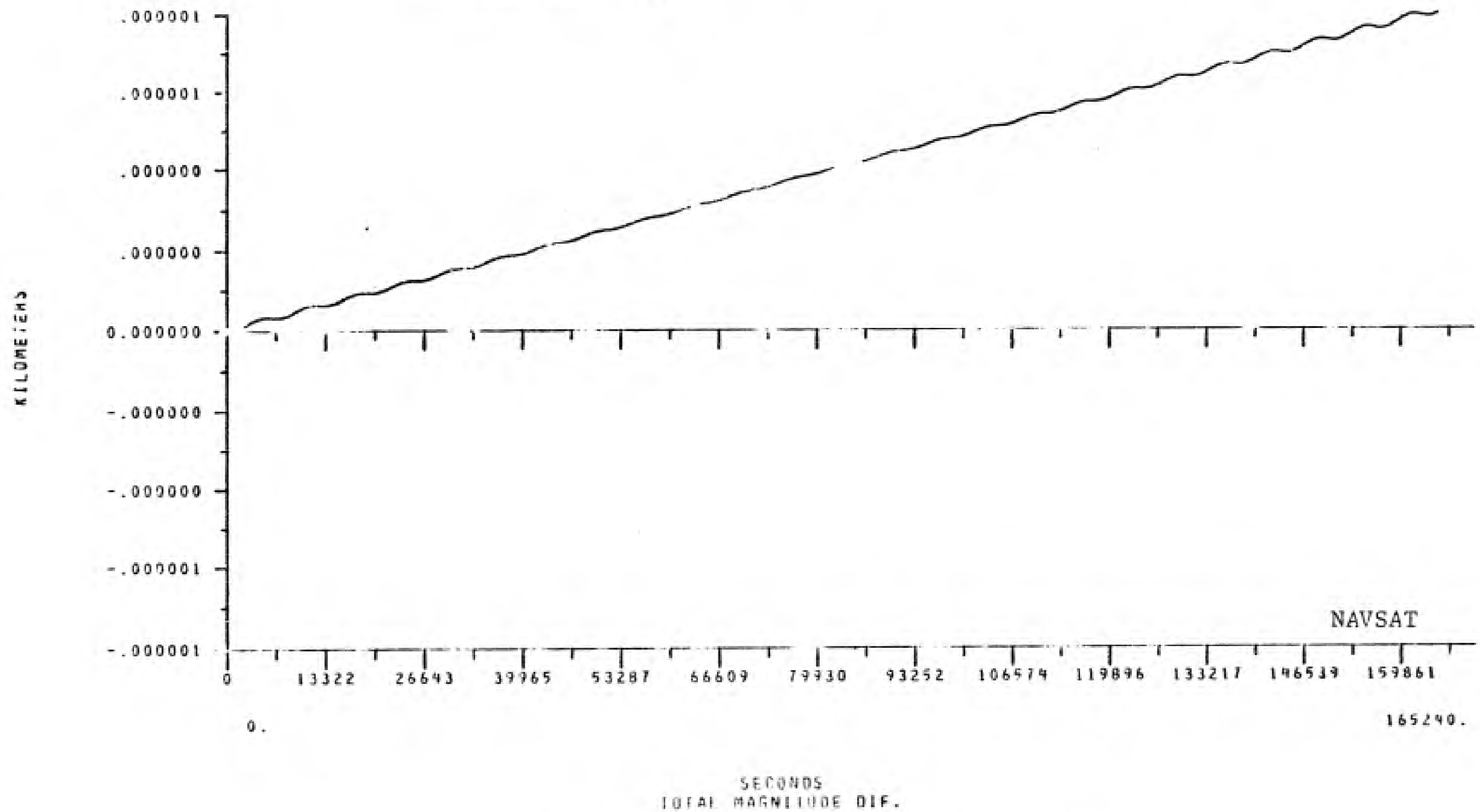
FIGURE 2

TRAJECTORY COMPARISON PLOT

REV NO. 0 IS COMPARED TO REV NO. 0

EPOCH (YR,DAY,SEC)= 74 193 3600

SATELLITE= 68



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1. Robinson, Jane V.,"Optimum Integration Interval for Use in Orbit Computations with the NWL-9 Gravitational Parameter Set," NWL Technical Report TR-2478.

2. Lehman, David J.,"Comparison of "Propogated" and "Integrated" Methods of Satellite Orbit Refinement", NWL Technical Report TR-3198.

APPENDIX A

MATHEMATICAL FORMULATION

Mathematical Formulation

The standard centered Lagrange interpolation formula is:

$$f(t) = \sum_{K=-m}^{m+1} A_K(s) f_K \qquad (1)$$

where:

$$A_K(s) = \frac{(-1)^{m-K+1}(s+m)^{[2m+2]}}{(m+K)!(m-K+1)!(s-K)} = \frac{P_{m,K}(s+m)^{[2m+2]}}{(s-K)} ,$$

$$(s+m)^{[2m+2]} = (s+m)(s+m-1)(s+m-2)\ldots(s+1)(s)(s-1)\ldots(s-m-1),$$

$$s = \frac{t-t_0}{\Delta t} \quad , \text{ and } \quad P_{m,K} = \frac{(-1)^{m-K+1}}{(m+K)!(m-K+1)!} .$$

To avoid loss of accuracy due to a zero divisor as $s \to 0$ or $s \to 1$, Equation 1 is rewritten as:

$$f(t) = (s+m)^{[2m+2]} \sum_{\substack{K=-m \\ K \neq 0,1}}^{m+1} \frac{P_{m,K}}{s-K} f_K + (s+m)_{0,1}^{[2m+2]} \left[P_{m,0}(s-1)f_0 + P_{m,1} s f_1 \right] \qquad (2)$$

where $(s+m)_{0,1}^{[2m+2]} = (s+m)\ldots(s+1)(s-2)(s-3)\ldots(s-m-1)$.

For the 1^{st} derivative we differentiate (2) and obtain

$$\dot{f}(t) = \frac{1}{\Delta t} \left\{ \sum_{\substack{K=-m \\ K \neq 0,1}}^{m+1} \frac{P_{m,K}}{(s-K)} \left[S - \frac{(s+m)^{[2m+2]}}{(s-K)} \right] f_K \right.$$

$$\left. + (s+m)_{0,1}^{[2m+2]} \left[P_{m,0} f_0 (1+(s-1)P) + P_{m,1} f_1 (1+sP) \right] \right\} \qquad (3)$$

where $S = (s+m)^{[2m+2]} P + (s+m)_{0,1}^{[2m+2]} (2s-1)$ and $P = \sum_{\substack{K=-m \\ K \neq 0,1}}^{m+1} \frac{1}{(s-K)}$.

The second derivative is found by differentiating (3). The result is

$$\ddot{f}(t) = \frac{1}{\Delta t^2} \left[\sum_{\substack{K=-m \\ K \neq 0,1}}^{m+1} P_{m,K} \dot{P} \left(S - \frac{(s+m)^{[2m+2]}}{s-K} \right. \right. +$$

$$\left. \frac{P_{m,K}}{s-K} \left(SP + \dot{P}(s+m)^{[2m+2]} + (s+m)_{0,1}^{[2m+2]} \left| P(2s-1)+2 \right| - \frac{(s+m)^{[2m+2]}}{(s-K)^2} + \frac{S}{(s-K)} \right) \right) f_K$$

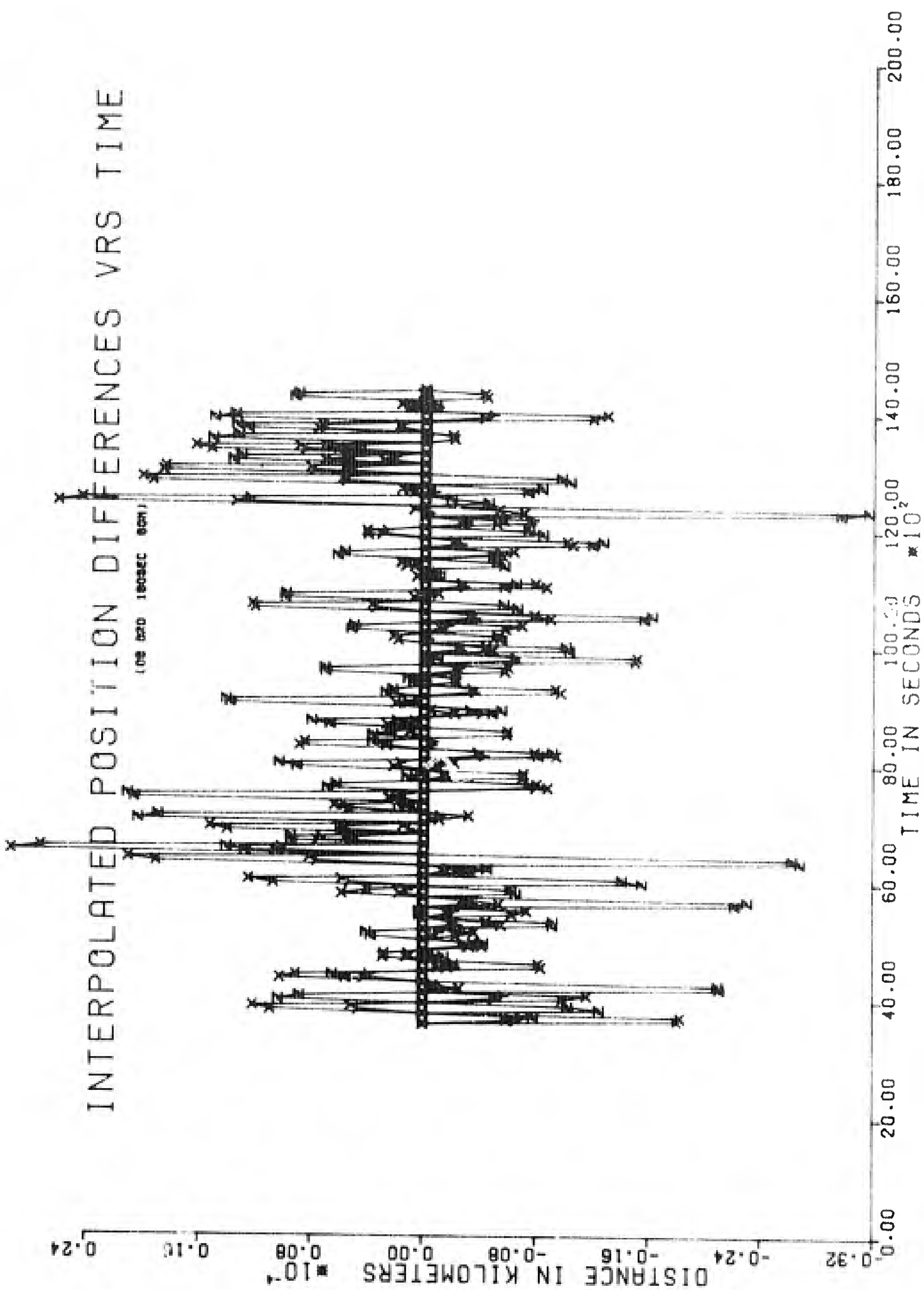
$$+ S \left[P_{m,0} f \, (1+(s-1)P) + P_{m,1} f_1 (1+sP) \right]$$

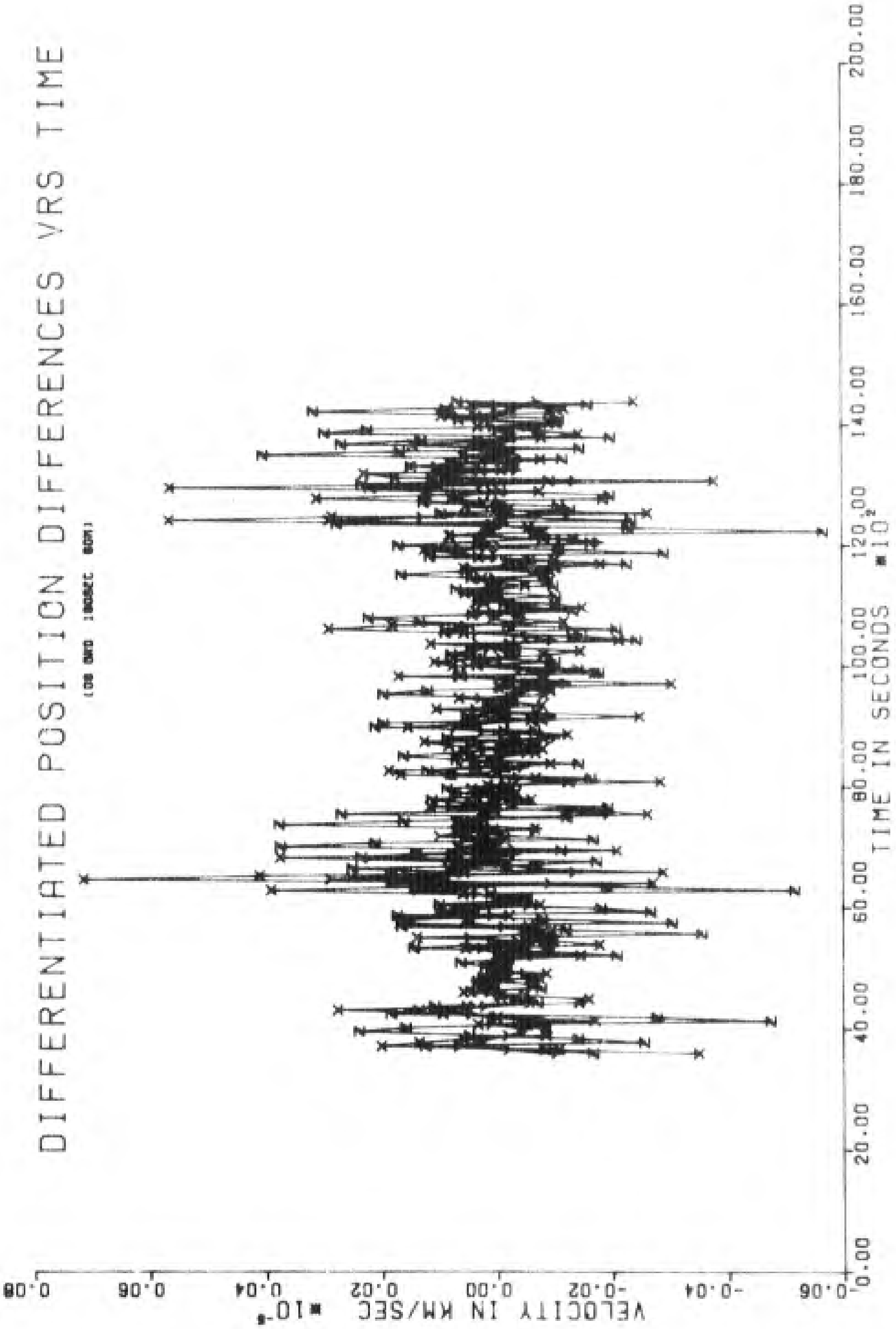
$$\left. + (s+m)_{0,1}^{[2m+2]} \left[P_{m,0} f_0 (P+s\dot{P}-\dot{P}) + P_{m,1} f_1 (P+s\dot{P}) \right] \right]$$

where $\dot{P} = \sum_{\substack{K=-m \\ K \neq 0,1}}^{m+1} \frac{1}{(s-K)^2}$.

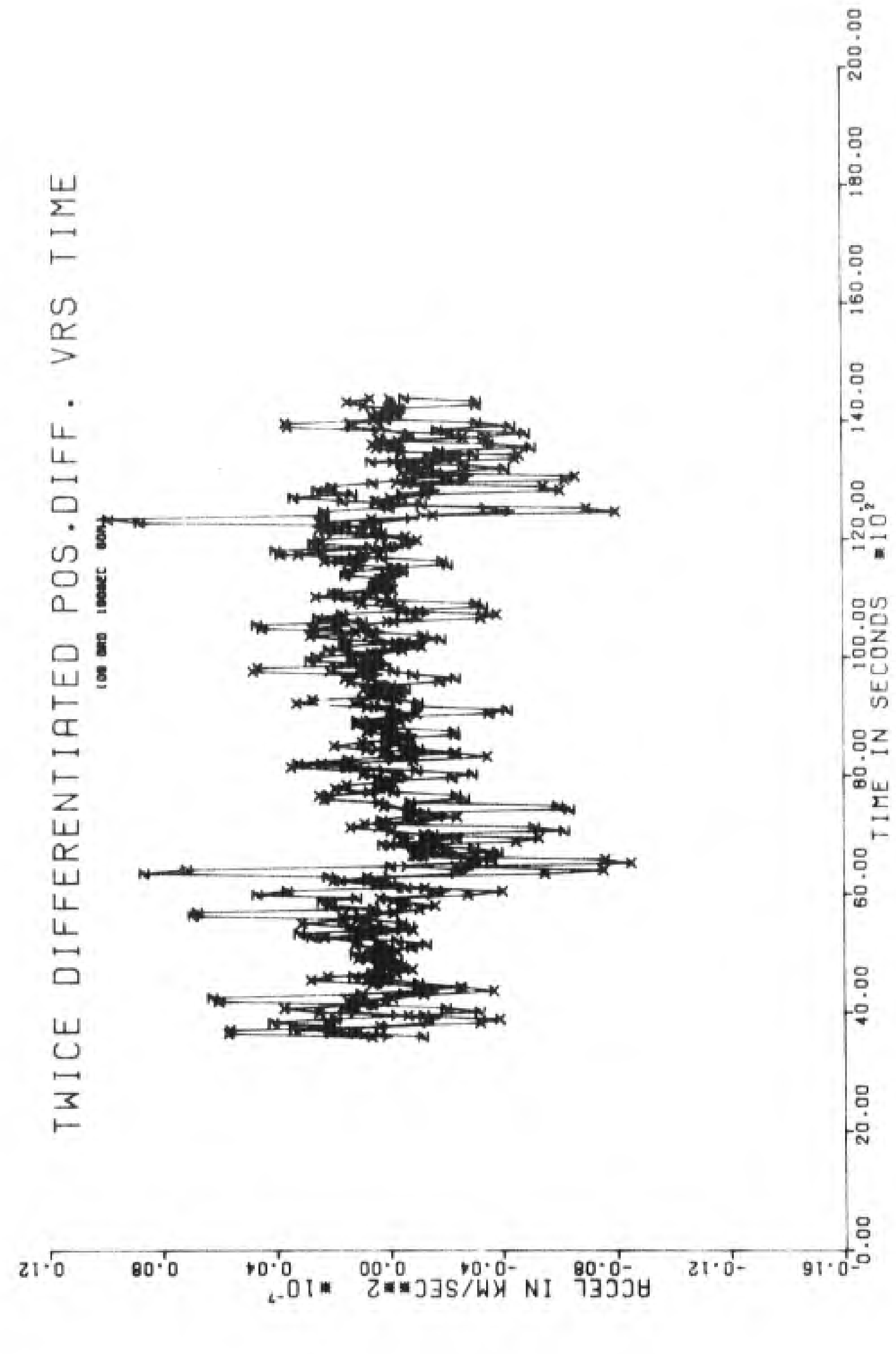
APPENDIX B

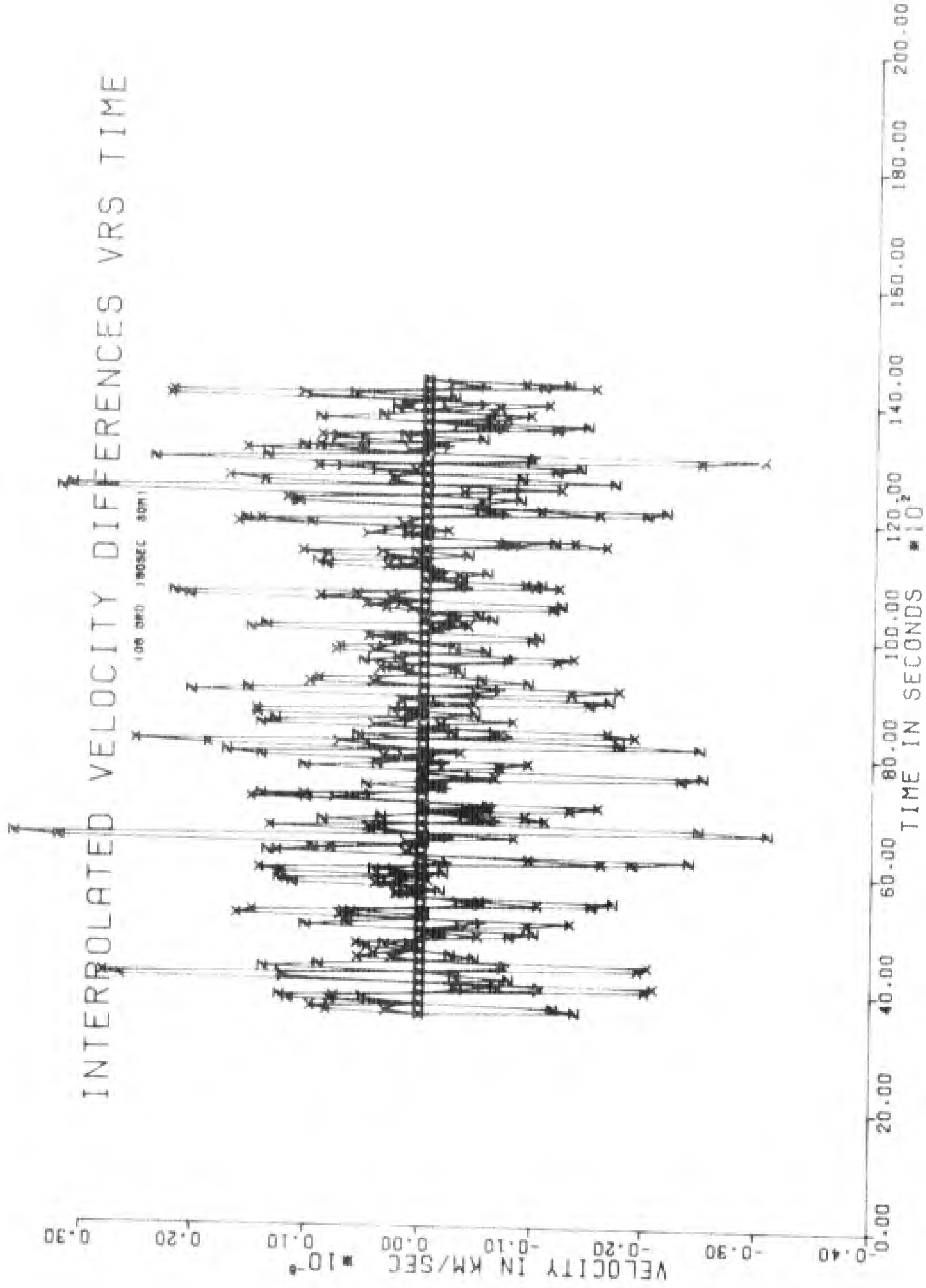
PLOTS OF RESULTING DIFFERENCES

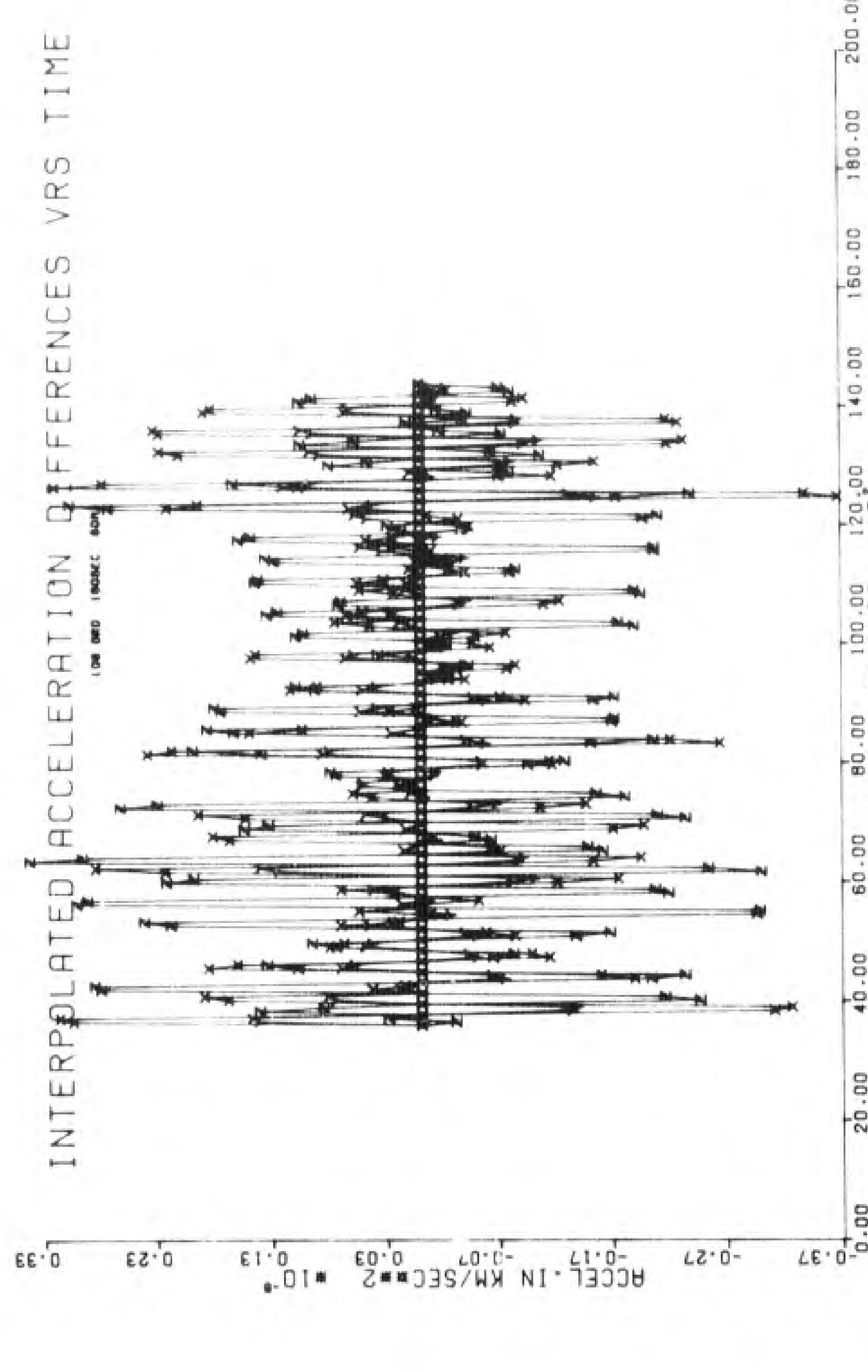

INTERPOLATED POSITION DIFFERENCES VRS TIME
TIME IN SECONDS *10²
DISTANCE IN KILOMETERS *10¹
0.00
20.00
40.00
60.00
80.00
100.00
120.00
140.00
160.00
180.00
200.00
-0.32
-0.24
-0.16
-0.08
0.00
0.08
0.16
0.24

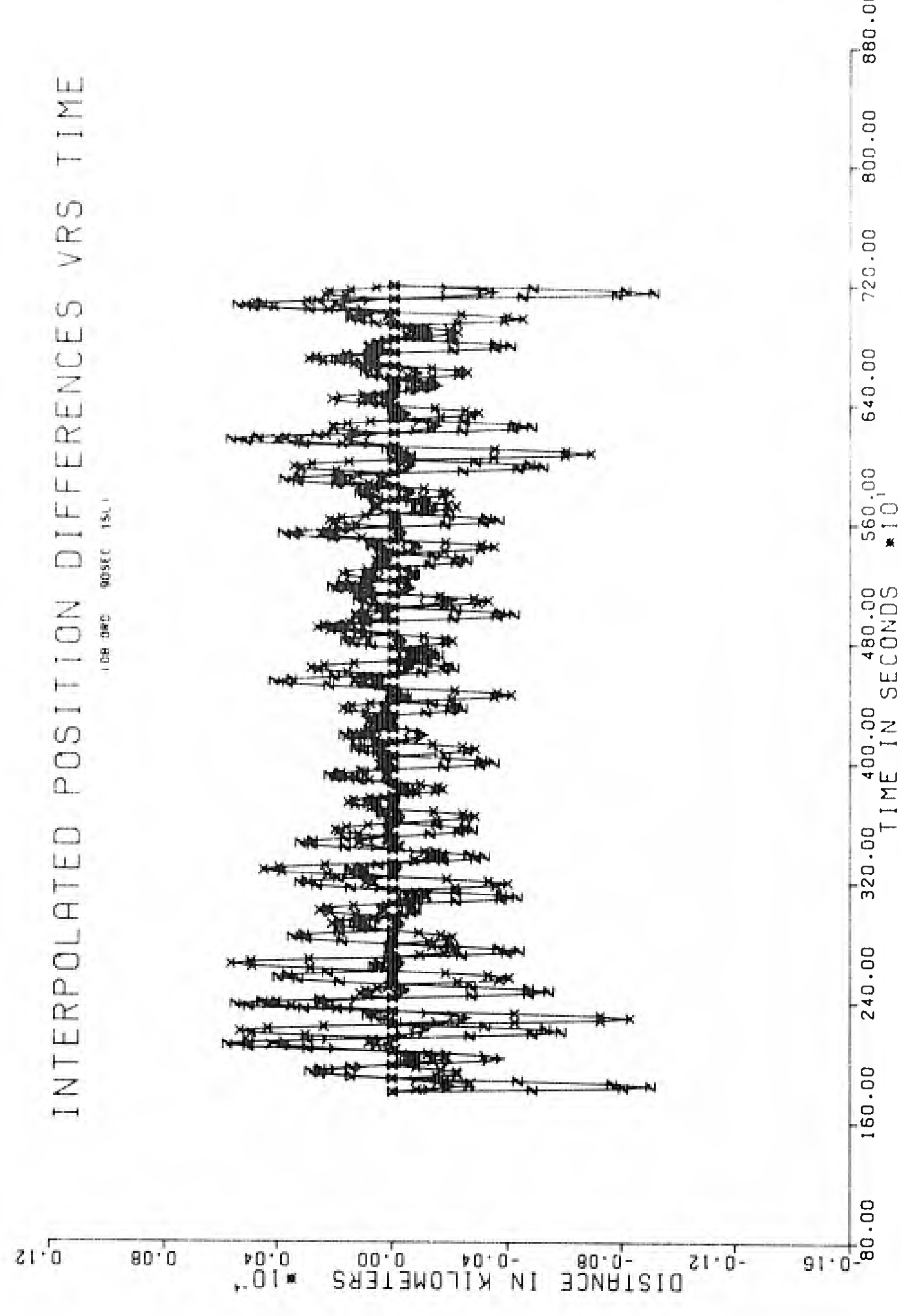


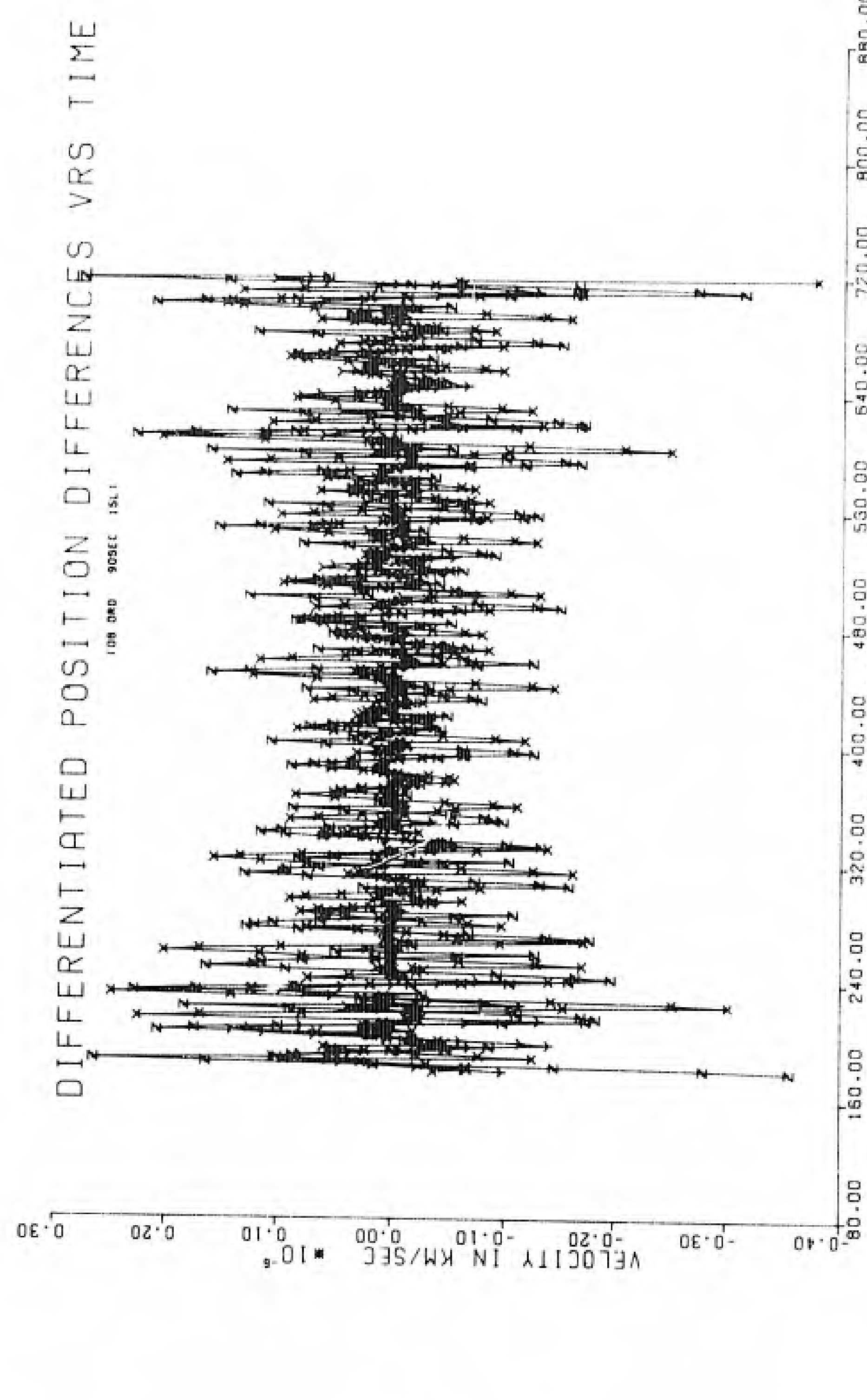
DIFFERENTIATED POSITION DIFFERENCES VRS TIME
TIME IN SECONDS
VELOCITY IN KM/SEC
0.00
20.00
40.00
60.00
80.00
100.00
120.00
140.00
160.00
180.00
200.00
-0.06
-0.04
-0.02
0.00
0.02
0.04
0.06
0.08


TWICE DIFFERENTIATED POS.DIFF. VRS TIME
TIME IN SECONDS *10^2
ACCEL IN KM/SEC**2 *10^-7
0.00
20.00
40.00
60.00
80.00
100.00
120.00
140.00
160.00
180.00
200.00
0.12
0.08
0.04
0.00
-0.04
-0.08
-0.12
-0.16

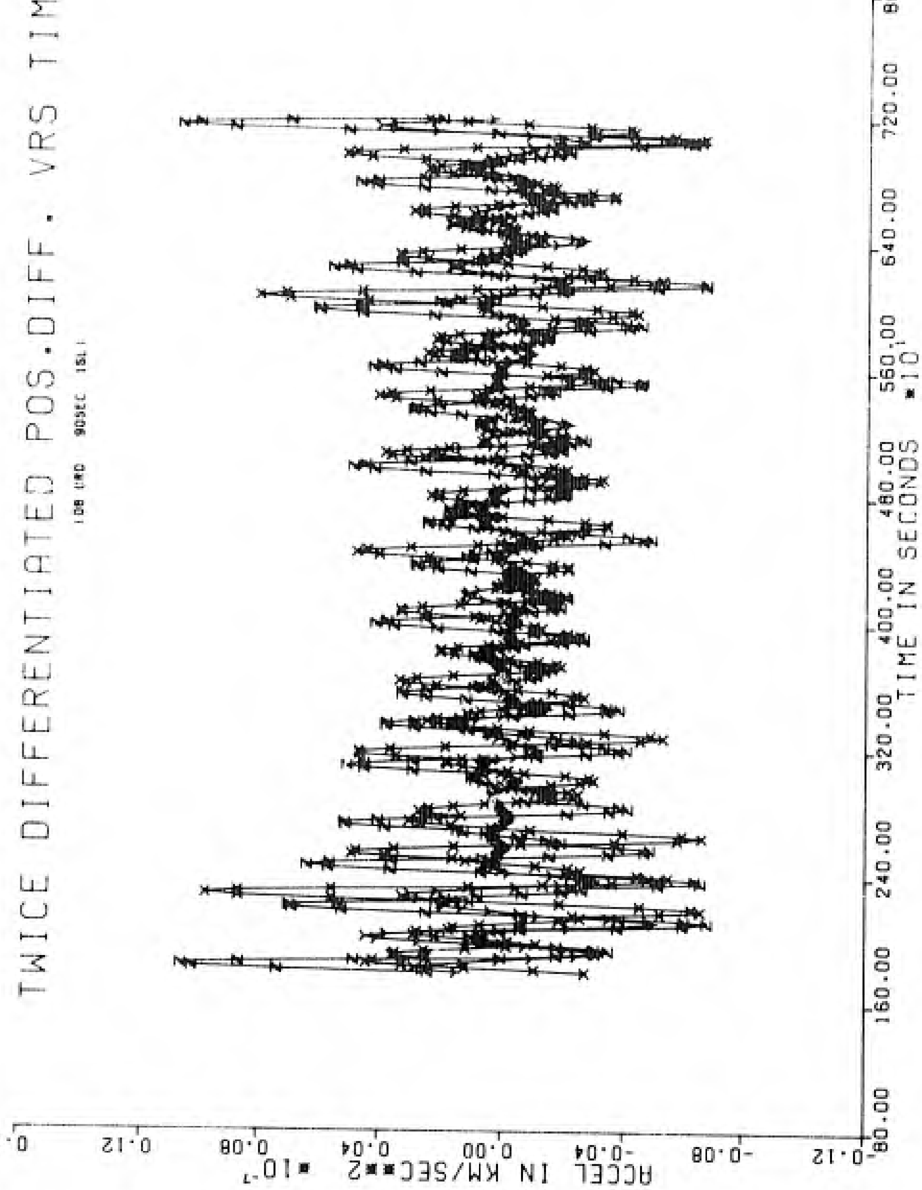

INTERPOLATED VELOCITY DIFFERENCES VRS TIME
TIME IN SECONDS *10^2
VELOCITY IN KM/SEC
0.00
20.00
40.00
60.00
80.00
100.00
120.00
140.00
160.00
180.00
200.00
0.30
0.20
0.10
0.00
-0.10
-0.20
-0.30
-0.40

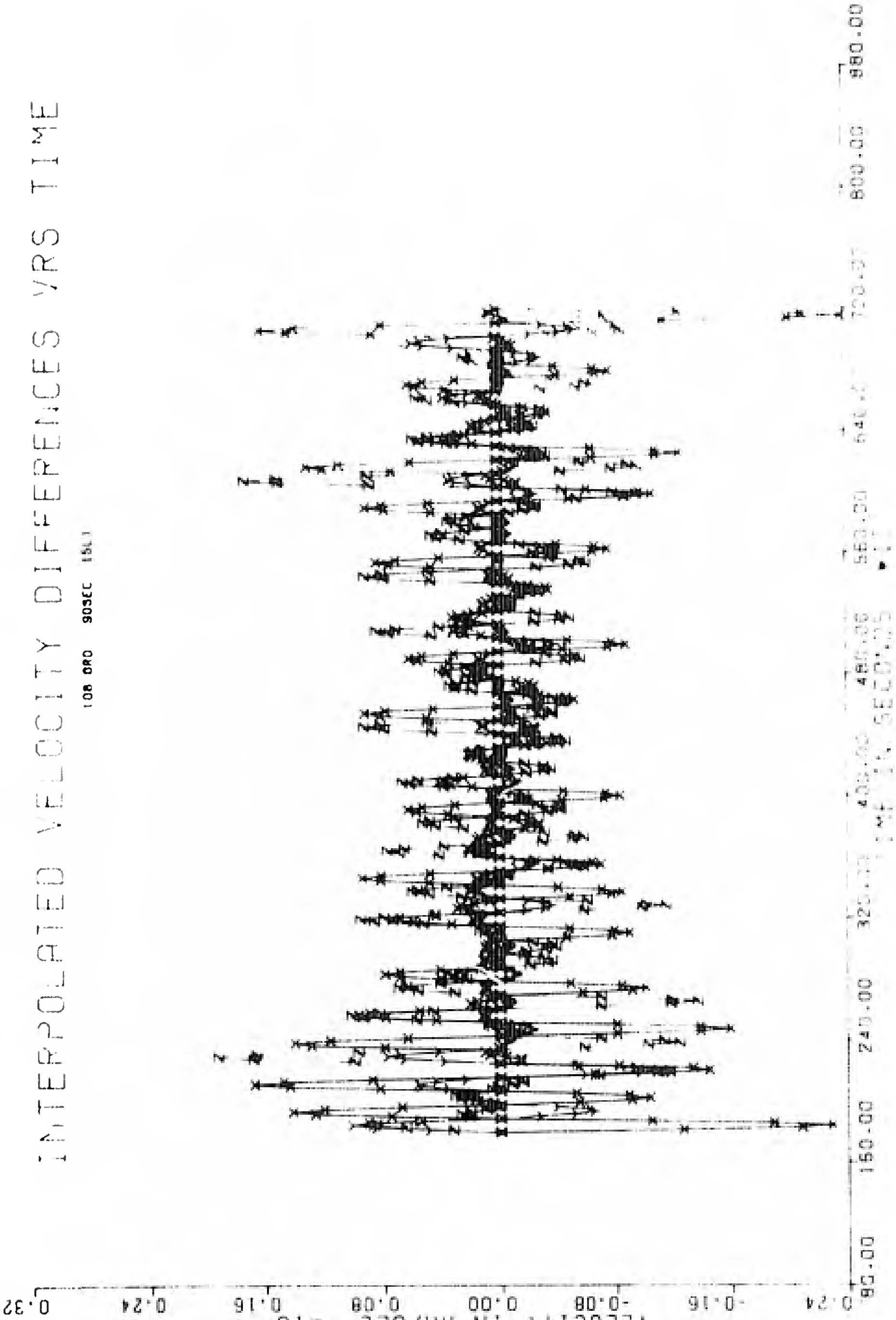

INTERPOLATED ACCELERATION DIFFERENCES VRS TIME
TIME IN SECONDS *10²
ACCEL. IN KM/SEC**2 *10
0.00
20.00
40.00
60.00
80.00
100.00
120.00
140.00
160.00
180.00
200.00
0.33
0.23
0.13
0.03
-0.07
-0.17
-0.27
-0.37

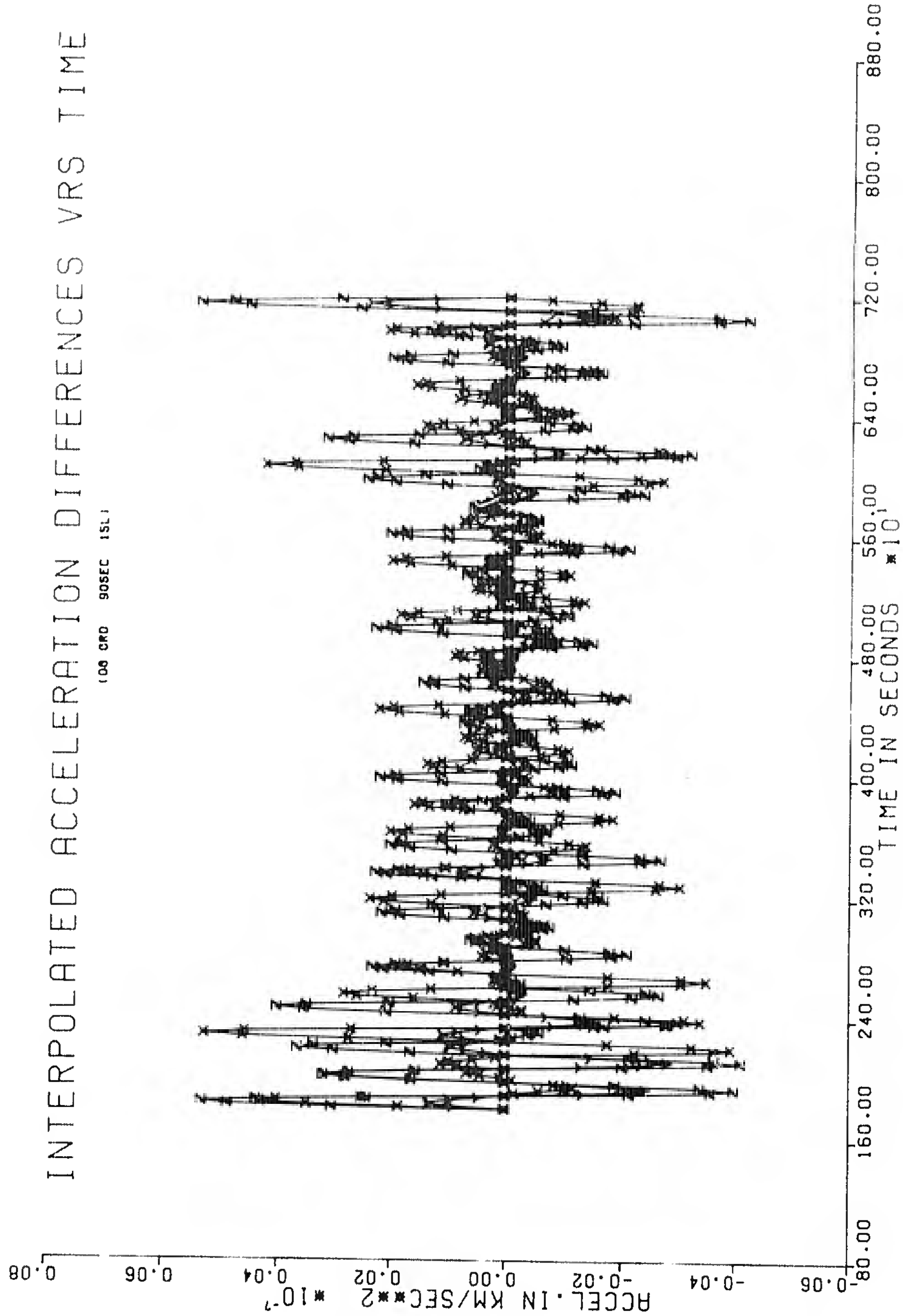

INTERPOLATED POSITION DIFFERENCES VRS TIME
TIME IN SECONDS *10^1
DISTANCE IN KILOMETERS *10^4

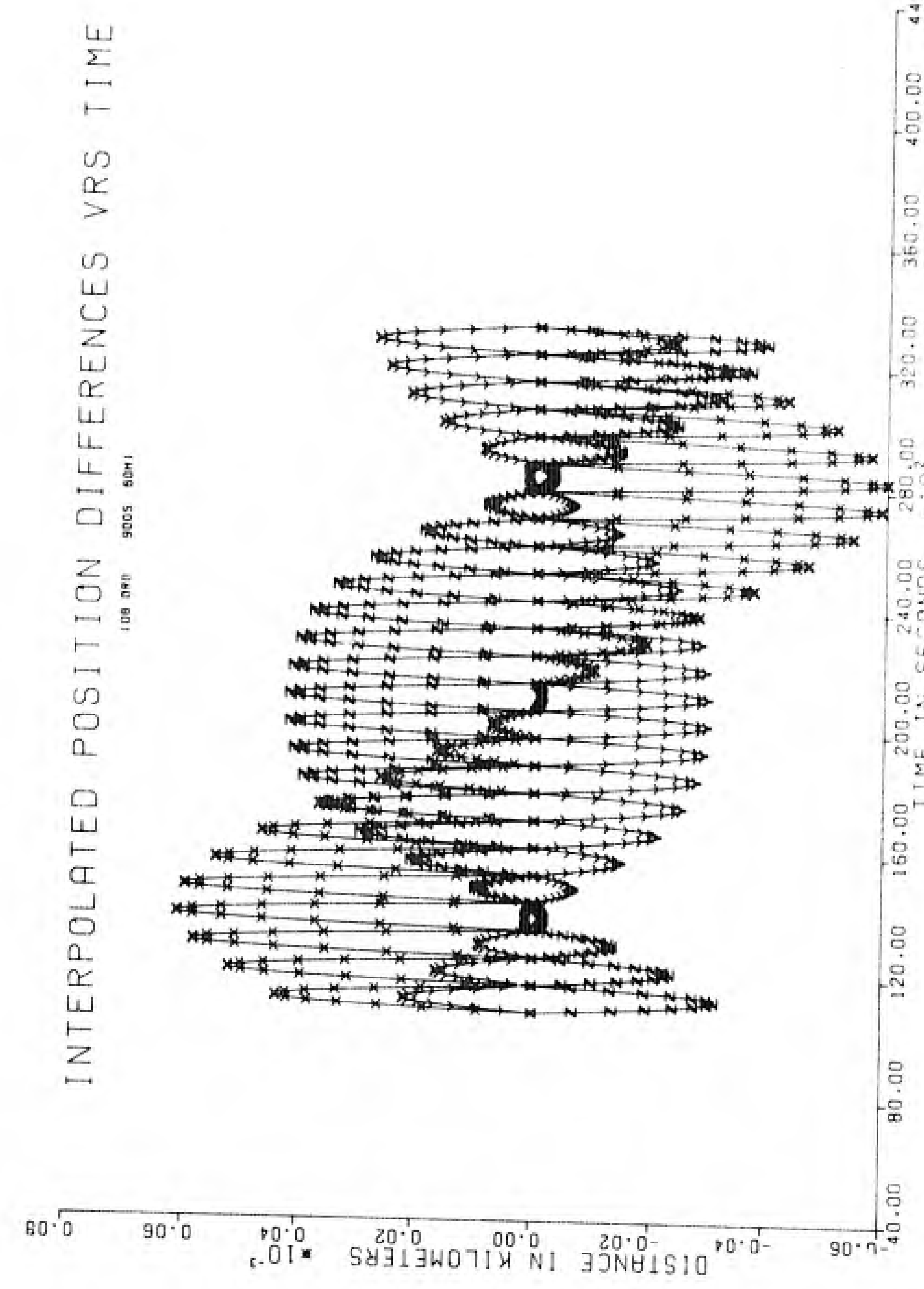


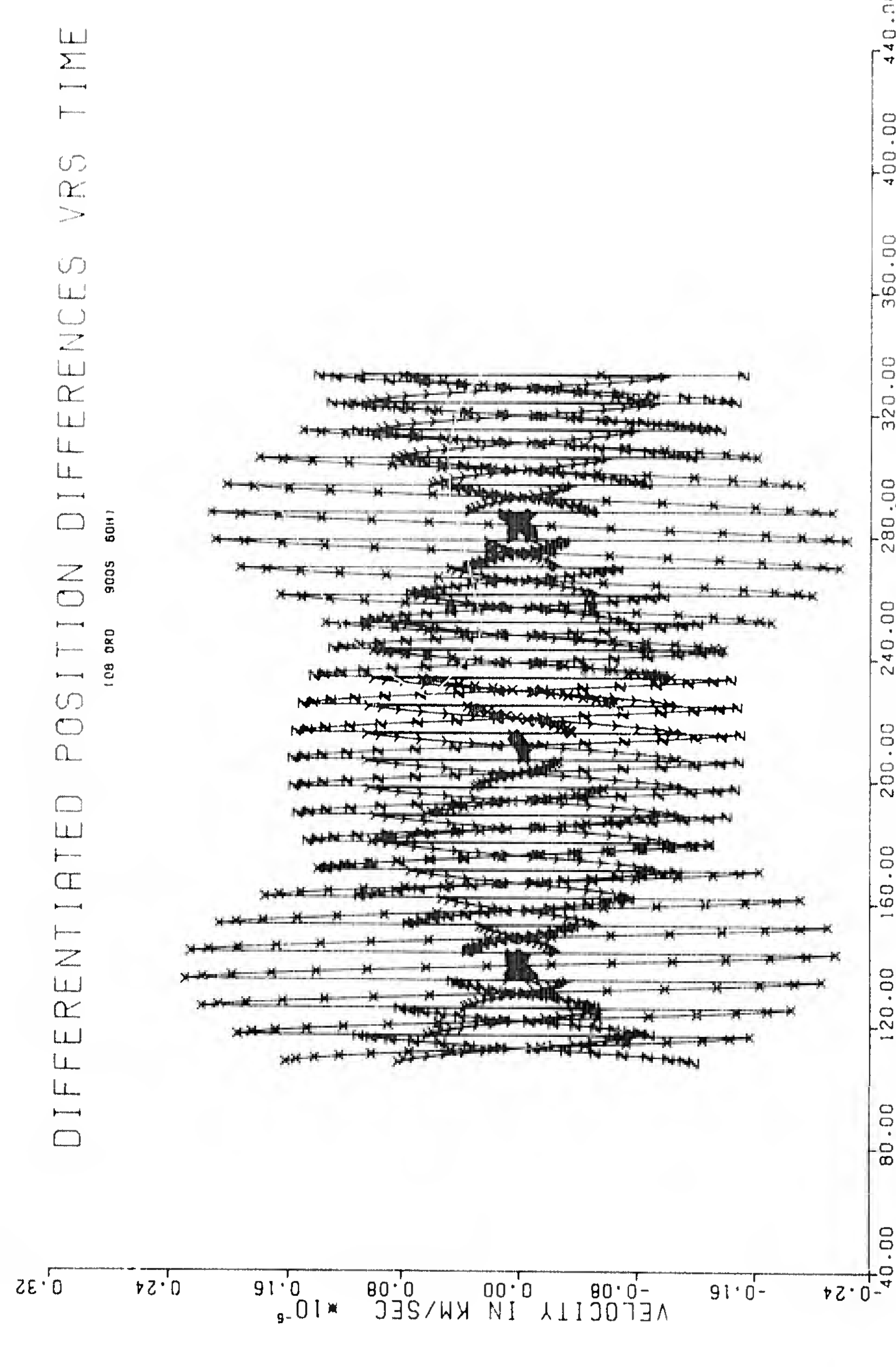
DIFFERENTIATED POSITION DIFFERENCES VRS TIME
TIME IN SECONDS *10^1
VELOCITY IN KM/SEC *10^-6
80.00
160.00
240.00
320.00
400.00
480.00
560.00
640.00
720.00
800.00
880.00
0.30
0.20
0.10
0.00
-0.10
-0.20
-0.30
-0.40

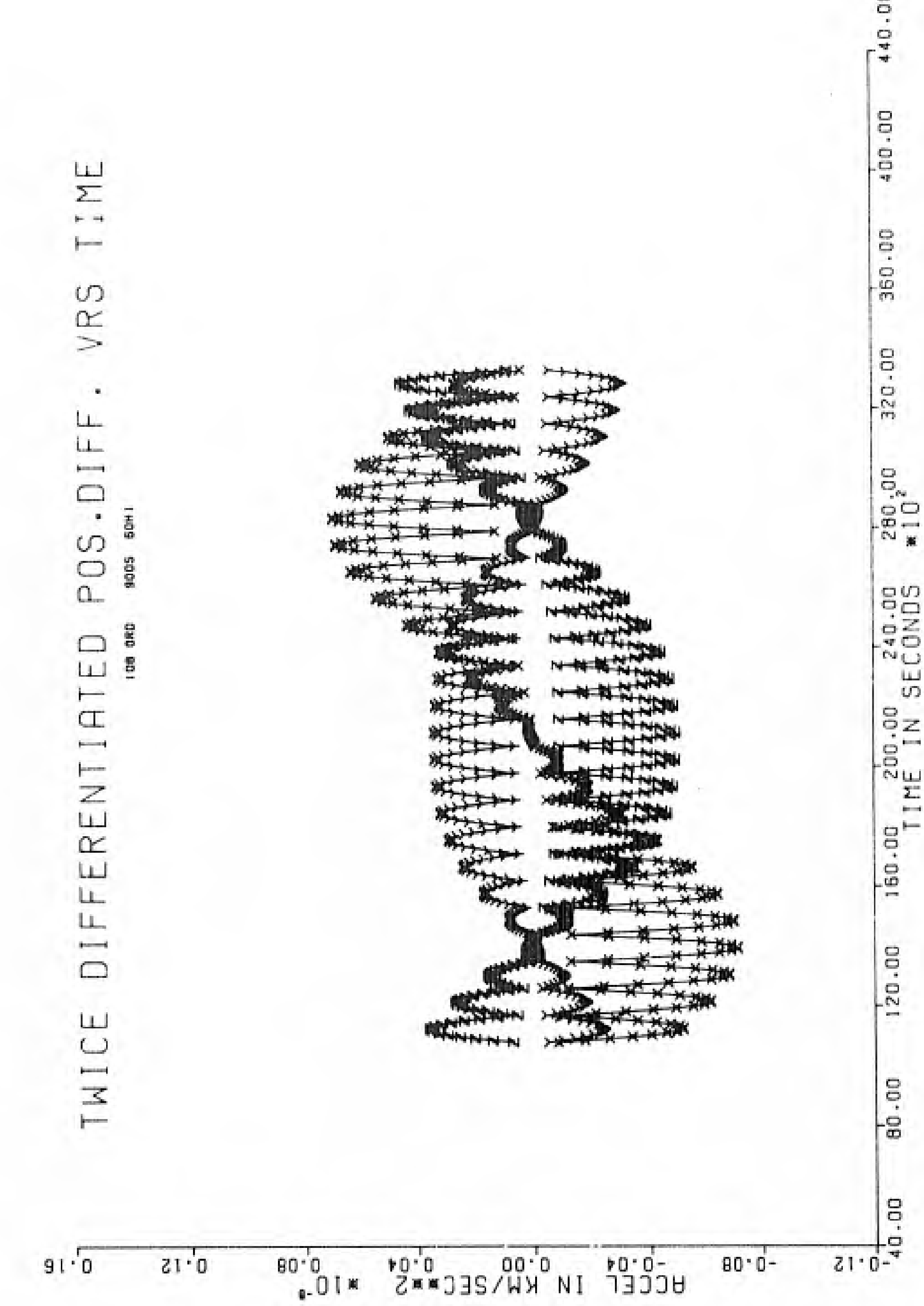

TWICE DIFFERENTIATED POS.DIFF. VRS TIME
TIME IN SECONDS *10^1
80.00
160.00
240.00
320.00
400.00
480.00
560.00
640.00
720.00
800.00
880.00
ACCEL IN KM/SEC**2 *10^-7
-0.12
-0.08
-0.04
0.00
0.04
0.08
0.12
0.16


INTERPOLATED VELOCITY DIFFERENCES VRS TIME
VELOCITY IN KM/SEC
0.32
0.24
0.16
0.08
0.00
-0.08
-0.16
-0.24
80.00
160.00
240.00
800.00
880.00

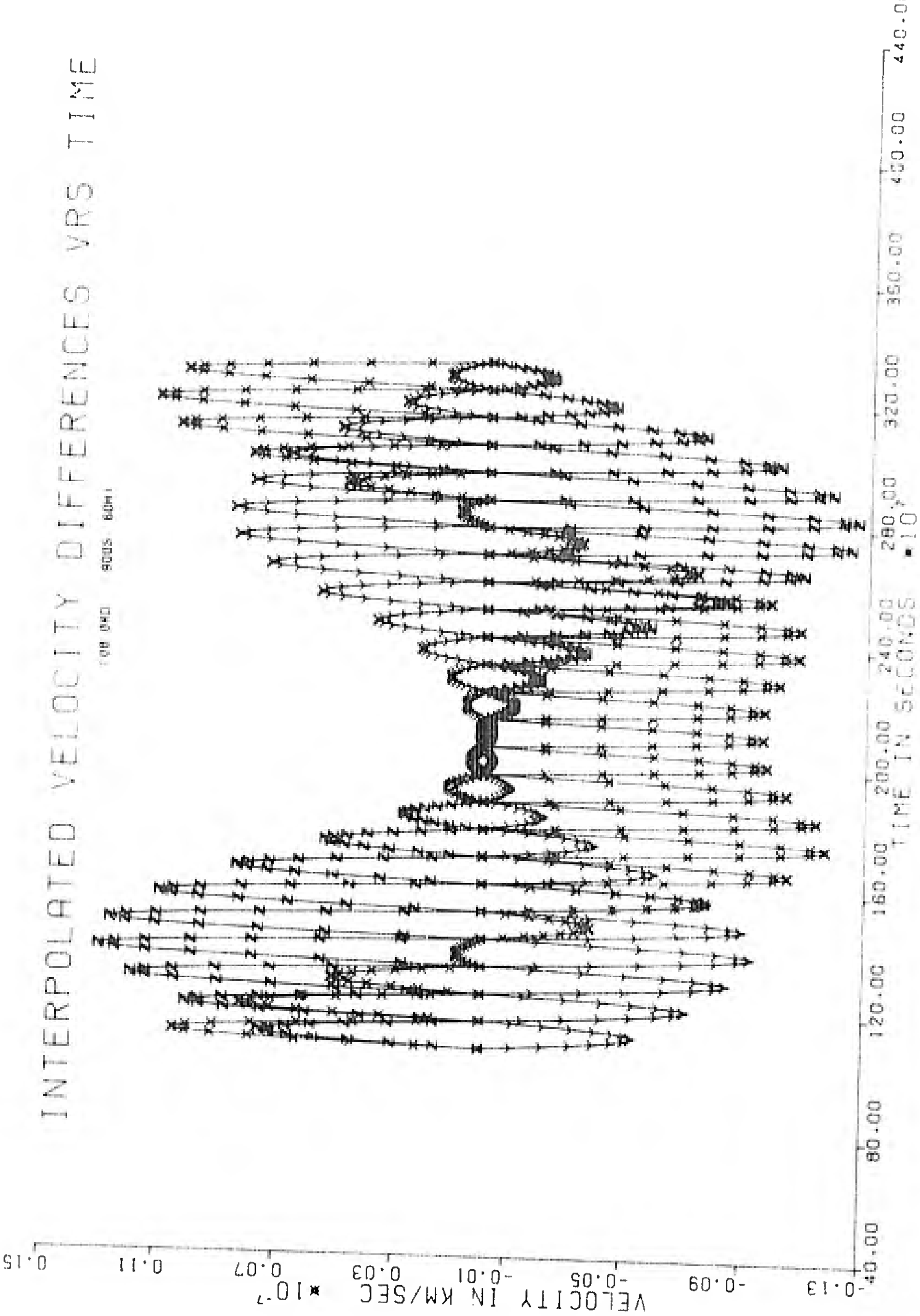

INTERPOLATED ACCELERATION DIFFERENCES VRS TIME
TIME IN SECONDS *10
ACCEL. IN KM/SEC**2 *10
80.00
160.00
240.00
320.00
400.00
480.00
560.00
640.00
720.00
800.00
880.00
-0.06
-0.04
-0.02
0.00
0.02
0.04
0.06
0.08

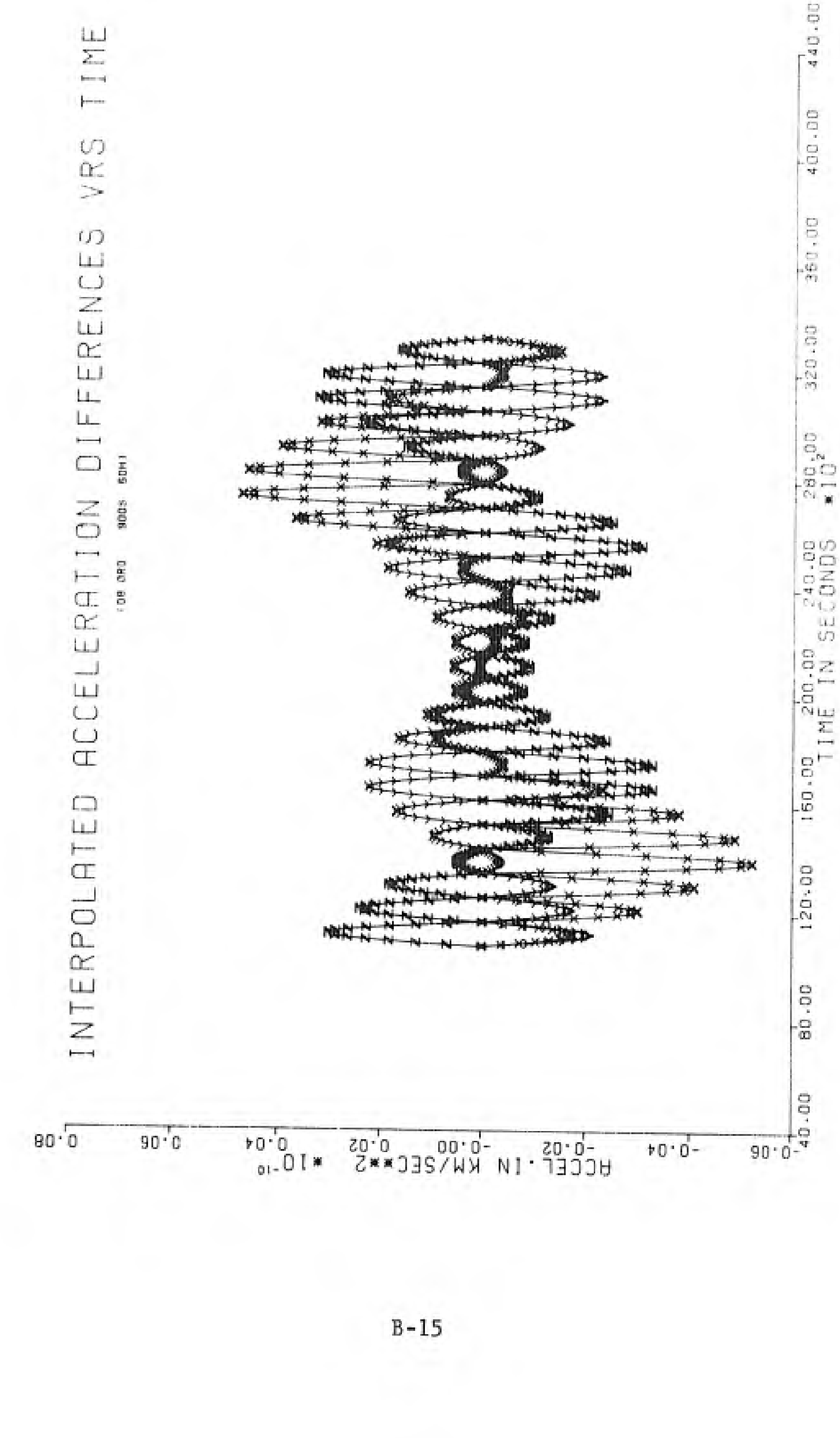

INTERPOLATED POSITION DIFFERENCES VRS TIME
TIME IN SECONDS *10²
DISTANCE IN KILOMETERS *10⁻³
40.00
80.00
120.00
160.00
200.00
240.00
280.00
320.00
360.00
400.00
440.00
-0.06
-0.04
-0.02
0.00
0.02
0.04
0.06
0.08


DIFFERENTIATED POSITION DIFFERENCES VRS TIME
108 ORD 9005 6011
TIME IN SECONDS *10²
VELOCITY IN KM/SEC *10⁻⁶
40.00
80.00
120.00
160.00
200.00
240.00
280.00
320.00
360.00
400.00
440.00
-0.24
-0.16
-0.08
0.00
0.08
0.16
0.24
0.32



TWICE DIFFERENTIATED POS.DIFF. VRS TIME
TIME IN SECONDS *10^2
ACCEL IN KM/SEC**2
40.00
80.00
120.00
160.00
200.00
240.00
280.00
320.00
360.00
400.00
440.00
-0.12
-0.08
-0.04
0.00
0.04
0.08
0.12
0.16


INTERPOLATED VELOCITY DIFFERENCES VRS TIME
TIME IN SECONDS
VELOCITY IN KM/SEC
80.00
120.00
160.00
200.00
240.00
280.00
320.00
360.00
400.00
440.00
-0.13
-0.09
-0.05
-0.01
0.03
0.07
0.11
0.15


INTERPOLATED ACCELERATION DIFFERENCES VRS TIME
TIME IN SECONDS *10^2
ACCEL. IN KM/SEC**2 *10^-10
40.00
80.00
120.00
160.00
200.00
240.00
280.00
320.00
360.00
400.00
440.00
-0.06
-0.04
-0.02
-0.00
0.02
0.04
0.06
0.08